

Relativistic Equations for Spin Particles: What Can We Learn From Noncommutativity?

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Abstract. We derive relativistic equations for charged and neutral spin particles. The approach for higher-spin particles is based on generalizations of the Bargmann-Wigner formalism. Next, we study, what new physical information can the introduction of non-commutativity give us. Additional non-commutative parameters can provide a suitable basis for explanation of the origin of mass.

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In the spin-1/2 case the Klein-Gordon equation can be written for the two-component spinor ($c = \hbar = 1$)

$$(EI^{(2)} - \boldsymbol{\sigma} \cdot \mathbf{p})(EI^{(2)} + \boldsymbol{\sigma} \cdot \mathbf{p})\Psi^{(2)} = m^2\Psi^{(2)}, \quad (1)$$

or, in the 4-component form

$$[i\gamma_\mu \partial_\mu + m_1 + m_2\gamma^5]\Psi^{(4)} = 0. \quad (2)$$

There exist various generalizations of the Dirac formalism, see [1, 2] and references therein. In the higher spin cases we can proceed in a similar way to obtain relativistic equations. On this basis we are ready to generalize the BW formalism [3]. Why is that convenient? In Ref. [4] the mapping has been presented between the Weinberg-Tucker-Hammer (WTH) equation, Ref. [5], and the equations for antisymmetric tensor fields (AST).

$$[\gamma_{\alpha\beta} p_\alpha p_\beta + A p_\alpha p_\alpha + B m^2]\Psi^{(6)} = 0, \quad (3)$$

which would give many relativistic equations for the AST field differing from the Proca theory.

We tried to find relations between the generalized WTH theory and other spin-1 formalisms. Therefore, we were forced to modify the Bargmann-Wigner formalism [6], which as has been claimed, does not deal with the parity discrete symmetry. For instance, we introduced the sign operator ε_i in the Dirac equations which are the input for the formalism for the symmetric 2-rank spinor:

$$[i\gamma_\mu \partial_\mu + \varepsilon_1 m_1 + \varepsilon_2 m_2 \gamma_5]_{\alpha\beta} \Psi_{\beta\gamma} = 0, \quad (4)$$

$$[i\gamma_\mu \partial_\mu + \varepsilon_3 m_1 + \varepsilon_4 m_2 \gamma_5]_{\gamma\beta} \Psi_{\alpha\beta} = 0, \quad (5)$$

In general we have 16 possible combinations, but 4 of them give the same sets of the Proca-like equations. We obtain [6]:

$$\partial_\mu A_\lambda - \partial_\lambda A_\mu + 2m_1 A_1 F_{\mu\lambda} + im_2 A_2 \varepsilon_{\alpha\beta\mu\lambda} F_{\alpha\beta} = 0, \quad (6)$$

$$\partial_\lambda F_{\mu\lambda} - \frac{m_1}{2} A_1 A_\mu - \frac{m_2}{2} B_2 \tilde{A}_\mu = 0, \quad (7)$$

with $A_1 = (\varepsilon_1 + \varepsilon_3)/2$, $A_2 = (\varepsilon_2 + \varepsilon_4)/2$, $B_1 = (\varepsilon_1 - \varepsilon_3)/2$, and $B_2 = (\varepsilon_2 - \varepsilon_4)/2$. See the additional constraints in the cited papers. So, we have the dual tensor and the pseudovector potential in the Proca-like sets. The pseudovector potential is the same as that which enters in the Duffin-Kemmer set for the spin 0. Moreover, it appears that the properties of the polarization vectors with respect to parity operation depend on the choice of the spin basis. For instance, in Ref. [6, 7] the momentum-space polarization vectors have been listed in the helicity basis. Berestetskiĭ, Lifshitz and Pitaevskiĭ claimed too, Ref. [8], that the helicity states cannot be the parity states. If one applies common-used relations between fields and potentials it appears that the \mathbf{E} and \mathbf{B} fields have no usual properties with respect to the space inversion.

Next, we developed the theory of the 4-vector field in the matrix form, including the spin-0 state [9, 10]. S. I. Kruglov proposed, Ref. [11], a general form of the Lagrangian for 4-potential field B_μ . We have

$$\alpha \partial_\mu \partial_\nu B_\nu + \beta \partial_\nu^2 B_\mu + \gamma m^2 B_\mu = 0, \quad (8)$$

provided that derivatives commute. When $\partial_\nu B_\nu = 0$ (the Lorentz gauge) we obtain spin-1 states only. However, if it is not equal to zero we have a scalar field and an pseudovector potential. The consistent theory is, in fact, a generalization of the Stueckelberg formalism [12].

The spin-2 case has also been considered in a similar way [13]. We begin with the equations for the 4-rank symmetric spinor:

$$[i\gamma^\mu \partial_\mu - m]_{\alpha\alpha'} \Psi_{\alpha'\beta\gamma\delta} = 0, [i\gamma^\mu \partial_\mu - m]_{\beta\beta'} \Psi_{\alpha\beta'\gamma\delta} = 0 \quad (9)$$

$$[i\gamma^\mu \partial_\mu - m]_{\gamma\gamma'} \Psi_{\alpha\beta\gamma'\delta} = 0, [i\gamma^\mu \partial_\mu - m]_{\delta\delta'} \Psi_{\alpha\beta\gamma\delta'} = 0. \quad (10)$$

The massless limit (if one needs) should be taken in the end of all calculations.

We proceed expanding the field function in the set of symmetric matrices (as in the spin-1 case). The total function is

$$\begin{aligned} \Psi_{\{\alpha\beta\}\{\gamma\delta\}} &= (\gamma_\mu R)_{\alpha\beta} (\gamma^\kappa R)_{\gamma\delta} G_\kappa{}^\mu + (\gamma_\mu R)_{\alpha\beta} (\sigma^{\kappa\tau} R)_{\gamma\delta} F_{\kappa\tau}{}^\mu + \\ &+ (\sigma_{\mu\nu} R)_{\alpha\beta} (\gamma^\kappa R)_{\gamma\delta} T_\kappa{}^{\mu\nu} + (\sigma_{\mu\nu} R)_{\alpha\beta} (\sigma^{\kappa\tau} R)_{\gamma\delta} R_{\kappa\tau}{}^{\mu\nu}; \end{aligned} \quad (11)$$

and the resulting tensor equations are:

$$\frac{2}{m} \partial_\mu T_\kappa{}^{\mu\nu} = -G_\kappa{}^\nu, \frac{2}{m} \partial_\mu R_{\kappa\tau}{}^{\mu\nu} = -F_{\kappa\tau}{}^\nu, \quad (12)$$

$$T_\kappa{}^{\mu\nu} = \frac{1}{2m} [\partial^\mu G_\kappa{}^\nu - \partial^\nu G_\kappa{}^\mu], \quad (13)$$

$$R_{\kappa\tau}{}^{\mu\nu} = \frac{1}{2m} [\partial^\mu F_{\kappa\tau}{}^\nu - \partial^\nu F_{\kappa\tau}{}^\mu]. \quad (14)$$

The constraints are re-written to

$$\frac{1}{m} \partial_\mu G_\kappa{}^\mu = 0, \quad \frac{1}{m} \partial_\mu F_{\kappa\tau}{}^\mu = 0, \quad (15)$$

$$\frac{1}{m}\varepsilon_{\alpha\beta\nu\mu}\partial^\alpha T_{\kappa}{}^{\beta\nu} = 0, \quad \frac{1}{m}\varepsilon_{\alpha\beta\nu\mu}\partial^\alpha R_{\kappa\tau}{}^{\beta\nu} = 0. \quad (16)$$

However, we need to make symmetrization over these two sets of indices $\{\alpha\beta\}$ and $\{\gamma\delta\}$. The total symmetry can be ensured if one contracts the function $\Psi_{\{\alpha\beta\}\{\gamma\delta\}}$ with *antisymmetric* matrices $R_{\beta\gamma}^{-1}$, $(R^{-1}\gamma^5)_{\beta\gamma}$ and $(R^{-1}\gamma^5\gamma^\lambda)_{\beta\gamma}$ and equate all these contractions to zero (similar to the $j = 3/2$ case considered in Ref. [3b,p.44]. We encountered with the known difficulty of the theory for spin-2 particles in the Minkowski space. We explicitly showed that all field functions become to be equal to zero. Such a situation cannot be considered as a satisfactory one (because it does not give us any physical information) and can be corrected in several ways. We modified the formalism [6]. The field function is now presented as

$$\Psi_{\{\alpha\beta\}\gamma\delta} = \alpha_1(\gamma_\mu R)_{\alpha\beta}\Psi_{\gamma\delta}^\mu + \alpha_2(\sigma_{\mu\nu}R)_{\alpha\beta}\Psi_{\gamma\delta}^{\mu\nu} + \alpha_3(\gamma^5\sigma_{\mu\nu}R)_{\alpha\beta}\tilde{\Psi}_{\gamma\delta}^{\mu\nu}. \quad (17)$$

The equations and constraints have been found between tensors of different parity properties[13].

The questions of "non-commutativity" see, for instance, in Ref. [14]. The assumption that operators of coordinates do *not* commute $[\hat{x}_\mu, \hat{x}_\nu]_- = i\theta_{\mu\nu}$ (or, alternatively, $[\hat{x}_\mu, \hat{x}_\nu]_- = iC_{\mu\nu}^\beta x_\beta$) has been first made by H. Snyder [15]. Later it was shown that such an ansatz may lead to non-locality. Thus, the Lorentz symmetry may be broken. On the other hand, the famous Feynman-Dyson proof of Maxwell equations [16] contains intrinsically the non-commutativity of velocities. While $[x^i, x^j]_- = 0$ therein, but $[\dot{x}^i(t), \dot{x}^j(t)]_- = \frac{i\hbar}{m^2}\varepsilon^{ijk}B_k \neq 0$ (at the same time with $[x^i, \dot{x}^j]_- = \frac{i\hbar}{m}\delta^{ij}$) that also may be considered as a contradiction with the well-accepted theories. Dyson wrote in a very clever way about this problem. Furthermore, it has recently been shown that notation and terminology, which physicists used when speaking about partial derivative of many-variables functions, are sometimes confusing [17]. The well-known physical example of the situation, when we have both explicite and implicate dependences of the function which derivatives act upon, is the field of an accelerated charge [18]. First, Landau and Lifshitz wrote that the functions depended on the retarded time t' and only through $t' + R(t')/c = t$ they depended implicitly on x, y, z, t . However, later they used the explicit dependence of R and fields on the space coordinates of the observation point too. Otherwise, the "simply" retarded fields do not satisfy the Maxwell equations. So, actually the fields and the potentials are the functions of the following forms: $A^\mu(x, y, z, t'(x, y, z, t))$, $\mathbf{E}(x, y, z, t'(x, y, z, t))$, $\mathbf{B}(x, y, z, t'(x, y, z, t))$.

Let us to work out one example in the momentum representation. In the general case of the "whole-partial" derivative one has

$$\frac{\hat{\partial}f(\mathbf{p}, E(\mathbf{p}))}{\hat{\partial}p_i} \equiv \frac{\partial f(\mathbf{p}, E(\mathbf{p}))}{\partial p_i} + \frac{\partial f(\mathbf{p}, E(\mathbf{p}))}{\partial E} \frac{\partial E}{\partial p_i}. \quad (18)$$

Applying this rule, we surprisingly find

$$[\frac{\hat{\partial}}{\hat{\partial}p_i}, \frac{\hat{\partial}}{\hat{\partial}E}]_- f(\mathbf{p}, E(\mathbf{p})) = -\frac{\partial f}{\partial E} \frac{\partial}{\partial E} (\frac{\partial E}{\partial p_i}). \quad (19)$$

We put forward the following ansatz in the momentum representation:

$$[\hat{x}^\mu, \hat{x}^\nu]_- = \omega(\mathbf{p}, E(\mathbf{p})) F_{||}^{\mu\nu} \frac{\partial}{\partial E}. \quad (20)$$

In the modern literature, the idea of the broken Lorentz invariance by this method is widely discussed. Let us turn now to the application of the presented ideas to the Dirac case. Recently, we analyzed Sakurai-van der Waerden method of derivations of the Dirac (and higher-spins too) equation. We can start from either the equation (1) or the 4-component equation

$$(EI^{(4)} + \alpha \cdot \mathbf{p} + m\beta)(EI^{(4)} - \alpha \cdot \mathbf{p} - m\beta)\Psi_{(4)} = 0. \quad (21)$$

We also postulate the non-commutativity

$$[E, \mathbf{p}^i]_- = \Theta^{0i} = \theta^i, \quad (22)$$

as usual. Therefore the equation (21) will *not* lead to the well-known equation $E^2 - \mathbf{p}^2 = m^2$. Instead, we have

$$\{E^2 - E(\alpha \cdot \mathbf{p}) + (\alpha \cdot \mathbf{p})E - \mathbf{p}^2 - m^2 - i\sigma \times I_{(2)}[\mathbf{p} \otimes \mathbf{p}]\} \Psi_{(4)} = 0 \quad (23)$$

For the sake of simplicity, we may assume the last term to be zero. Thus we come to

$$\{E^2 - \mathbf{p}^2 - m^2 - (\alpha \cdot \theta)\} \Psi_{(4)} = 0. \quad (24)$$

However, let us make the unitary transformation. It is known [19] that one can

$$U_1(\sigma \cdot \mathbf{a})U_1^{-1} = \sigma_3|\mathbf{a}|. \quad (25)$$

The final equation is

$$[E^2 - \mathbf{p}^2 - m^2 - \gamma_{chiral}^5 |\theta|] \Psi'_{(4)} = 0. \quad (26)$$

In the physical sense this implies the mass splitting for the Dirac particle over the non-commutative space. We have two solutions for $m_1 = \sqrt{m^2 + |\theta|}$ and $m_2 = \sqrt{m^2 - |\theta|}$. This procedure may be attractive for explanation of the mass creation and the mass splitting for fermions.

The conclusions are: 1) The $(1/2, 1/2)$ representation contains both the spin-1 and spin-0 states (cf. with the Stueckelberg formalism). 2) Unless we take into account the fourth state (the “time-like” state, or the spin-0 state) the set of 4-vectors is *not* a complete set in a mathematical sense. 3) We cannot remove terms like $(\partial_\mu B_\mu^*)(\partial_\nu B_\nu)$ terms from the Lagrangian and dynamical invariants unless apply the Fermi method, i. e., manually. The Lorentz condition applies only to the spin 1 states. 4) We have some additional terms in the expressions of the energy-momentum vector (and, accordingly, of the 4-current and the Pauli-Lunbanski vectors), which are the consequence of the impossibility to apply the Lorentz condition for spin-0 states. 5) Helicity vectors are not eigenvectors of the parity operator. Meanwhile, the parity is a “good” quantum number, $[\mathcal{P}, \mathcal{H}]_- = 0$ in the Fock space. 6) We are able to describe the states of different

masses in this representation from the beginning. 7) Various-type field operators can be constructed in the $(1/2, 1/2)$ representation space. For instance, they can contain C , P and CP conjugate states. Even if $b_\lambda^\dagger = a_\lambda^\dagger$ we can have complex 4-vector fields. We found the relations between creation, annihilation operators for different types of the field operators B_μ . 8) Propagators have good behaviour in the massless limit as opposed to those of the Proca theory. 9) The spin-2 case can be considered on an equal footing with the spin-1 case. 10) The postulate of non-commutativity leads to the mass splitting for leptons.

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